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A new integrable hierarchy of lattice equations

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Abstract. A new type of discrete spectral problem is introduced and studied. The related hierarchy of isospectral flows is tri-Hamiltonian. The Miura maps and modifications are presented.

1. Introduction

Recently, we generalized the results of [1, 2] on energy-dependent Schrödinger operators to third-order Lax operators [3], obtaining some generalizations of the well-known Boussinesq equation. The interesting thing is that, unlike the Schrödinger case, there exist only four possible extensions, since the second Hamiltonian operator for the Boussinesq equation is quadratic in the field variables. The purpose of this paper is to follow the ideas of [1-3] and to obtain some generalizations of interesting discrete equations. The first candidate is the well known Toda lattice because the structure of this system bears a great similarity to that of the Boussinesq equation. In fact, one of the continuum limits of the Toda lattice is the Boussinesq equation.

Recently, Tu [4] generalized his 'trace identity' approach to discrete integrable systems. By means of this method, he recovered the first Hamiltonian structure of the Toda lattice and its conserved quantities. The advantage of the method is its algorithmic nature. The disadvantage is that one does not always know how to derive the recursion operator.

The paper is organized as follows. In the next section, we calculate the Hamiltonian operators. Our Miura maps and modified systems will be presented in section 3.

2. Hamiltonian structures from a novel spectral problem

We consider the following spectral problem:

$$(E^{-1}+u+vE)\phi=0 \tag{2.1}$$

where $v=v_0+\lambda v_1$, $u=u_0+\lambda u_1$, and $Ef(n)=f(n+1)$, $E^{-1}f(n)=f(n-1)$. The Toda hierarchy corresponds to the case: $u_1=-1$ and $v_1=0$. As usual, we adjoin to (2.1) a time evolution equation of ϕ:

$$\phi_t=v(PE-Q)\phi. \tag{2.2}$$

The compatibility of (2.1) and (2.2) leads us to the following equation:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t=\begin{pmatrix} vE-E^{-1}v & u(I-E^{-1})v \\ v(E-I)u & v(E-E^{-1})v \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix}\equiv J\mathbb{P}. \tag{2.3}$$

Remarks. (1) We do not give the details of this calculation, which is straightforward. The choice of the form of the equation (2.2) is based on the work of Ragnisco [5] and Ragnisco and Santini [6].

(2) The operator J in (2.3) is easily recognized as the second Hamiltonian operator of Toda hierarchy [7]. In fact, if we put $u \to u - \lambda$, $v \to v$, we recover the first two Hamiltonian operators of the Toda lattice eqaution.

Now, with the expansions $u = u_0 + \lambda u_1$, $v = v_0 + \lambda v_1$, we have from (2.3)

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}_t + \lambda \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}_t = (J_0 + \lambda J_1 + \lambda^2 J_2)\mathbb{P} \tag{2.4}$$

where

$$J_0 = \begin{pmatrix} v_0 \boldsymbol{E} - \boldsymbol{E}^{-1} v_0 & u_0(\boldsymbol{I} - \boldsymbol{E}^{-1}) v_0 \\ v_0(\boldsymbol{E} - \boldsymbol{I}) u_0 & v_0(\boldsymbol{E} - \boldsymbol{E}^{-1}) v_0 \end{pmatrix}$$

$$J_1 = \begin{pmatrix} v_1 \boldsymbol{E} - \boldsymbol{E}^{-1} v_1 & u_0(\boldsymbol{I} - \boldsymbol{E}^{-1}) v_1 + u_1(\boldsymbol{I} - \boldsymbol{E}^{-1}) v_0 \\ v_0(\boldsymbol{E} - \boldsymbol{I}) u_1 + v_1(\boldsymbol{E} - \boldsymbol{I}) u_0 & v_0(\boldsymbol{E} - \boldsymbol{E}^{-1}) v_1 + v_1(\boldsymbol{E} - \boldsymbol{E}^{-1}) v_0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & u_1(\boldsymbol{I} - \boldsymbol{E}^{-1}) v_1 \\ v_1(\boldsymbol{E} - \boldsymbol{I}) u_1 & v_1(\boldsymbol{E} - \boldsymbol{E}^{-1}) v_1 \end{pmatrix}.$$

We can obtain the isospectral flows by means of the polynomial expansions for $\mathbb{P}$:

$$\mathbb{P}_{(m)} = \sum_{k=0}^{m} \mathbb{P}_{m-k} \lambda^k. \tag{2.5a}$$

Substituting (2.5a) into (2.4), we have the following the equations:

$$J_0 \mathbb{P}_{k-2} + J_1 \mathbb{P}_{k-1} + J_2 \mathbb{P}_k = 0 \qquad k = 0, \ldots, m-1 \tag{2.5b}$$

$$\begin{pmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{pmatrix}_t = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix} \begin{pmatrix} \mathbb{P}_{m-1} \\ \mathbb{P}_m \end{pmatrix} \tag{2.5c}$$

where $\boldsymbol{u}_k = (u_k, v_k)^T$. Here we need to solve for $\boldsymbol{P}_k$ recursively.

An infinite sequence of such polynomial expansions can be obtained by seeking a formal series solution

$$\mathscr{P} = \sum_{n=0}^{\infty} \mathbb{P}_n \lambda^{-n} \tag{2.6a}$$

of the equation $J\mathscr{P} = 0$. With such a solution, we may obtain an expansion for each $m \geqslant 1$

$$\mathbb{P}_{(m)} = (\lambda^m \mathscr{P})_+ \tag{2.6b}$$

where $(\ )_+$ denotes the truncation with only non-negative terms. The key step in this set-up is that by means of the solvability of the recursion relation (2.5b), the equations of the motion are equipped with the following three Hamiltonian forms:

$$\boldsymbol{u}_{t_m} = \boldsymbol{B}_2 \mathbb{P}^{(m)} = \boldsymbol{B}_1 \mathbb{P}^{(m+1)} = \boldsymbol{B}_0 \mathbb{P}^{(m+2)} \tag{2.7a}$$

where

$$\boldsymbol{B}_2 = \begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix} \qquad \boldsymbol{B}_1 = \begin{pmatrix} J_0 & 0 \\ 0 & -J_2 \end{pmatrix} \qquad \boldsymbol{B}_0 = \begin{pmatrix} -J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix} \tag{2.7b}$$

and

$$\mathbb{P}^{(m)} = (\mathbb{P}_{m-1}, \mathbb{P}_m)^T \qquad \boldsymbol{u} = (\boldsymbol{u}_0, \boldsymbol{u}_1)^T.$$

In order to make use of Magri's lemma [8, 9], we need to prove:

(1) $\boldsymbol{B}_0, \boldsymbol{B}_1, \boldsymbol{B}_2$ are compatible Hamiltonian operators.

(2) The equation $\boldsymbol{B}_0\mathbb{P}^{(m+1)} = \boldsymbol{B}_1\mathbb{P}^{(m)}$ has non-trivial solutions.

(3) There exist functions H_0 and H_1, such that $\boldsymbol{u}_t = \boldsymbol{B}_0\delta H_1 = \boldsymbol{B}_1\delta H_0$.

The proof of the assertion 1 will be given in the next section. Now, we prove the assertion 2. We notice that this is equivalent to the solvability of the equation $J\mathscr{P} = 0$. The argument used here is similar to those used by Schilling [10] and Tu [4] in the context of Ablowitz-Ladik and the Toda lattice respectively. To this end, we may rewrite the spectral problem (2.1) in the matrix form

$$\boldsymbol{E}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -v^{-1} & -v^{-1}u\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \equiv \mathbb{U}\Psi \tag{2.8a}$$

accordingly, the time evolution equation of ϕ (2.2) takes the following form:

$$\Psi_t = \mathbb{V}\Psi \qquad \mathbb{V} = \begin{pmatrix}-(vQ)^{(-1)} & (vP)^{(-1)}\\ -P & -uP - vQ\end{pmatrix}. \tag{2.8b}$$

Then, the stationary equation of the discrete zero-curvature is:

$$\mathbb{U}\mathbb{V} - (\boldsymbol{E}\mathbb{V})\mathbb{U} = 0 \tag{2.9}$$

which is exactly the equation $J\mathscr{P} = 0$.

As argued by Tu [4], the equation (2.9) implies

$$\mathrm{Trace}(\mathbb{V}) = \text{constant} \qquad \mathrm{Trace}(\mathbb{V}^2) = \text{constant}$$

or explicitly

$$(vQ)^{(-1)} + uP + vQ = C_1(\lambda) \tag{2.10a}$$

$$(vQ)^{(-1)}(uP + vQ) + (vP)^{(-1)}P = C_2(\lambda). \tag{2.10b'}$$

By means of (2.10a), (2.10b') may be rewritten as:

$$(vQ)(C_1(\lambda) - vQ) + vPP^{(1)} = C_2(\lambda). \tag{2.10b}$$

Recalling that $u = u_0 + \lambda u_1$, $v = v_0 + \lambda v_1$ and letting $P = \Sigma_{n\geqslant 0} P_n\lambda^{-n}$, $Q = \Sigma_{n\geqslant 0} Q_n\lambda^{-n}$, $C_1(\lambda) = c_1\lambda$, $C_2(\lambda) = c_2\lambda^2$, we can easily calculate the first few solutions listed below (2.11). Since the leading term in the coefficient of λ^{2-N} of the equation (2.10b) is $2v_1^2 Q_0 Q_N$, it is easy to see that we always have the local solutions P_i, Q_i, which are the solutions of the equation $J\mathscr{P} = 0$. Thus we have proved the solvability of equation $J\mathscr{P} = 0$.

Assertion 3 is proved by direct calculation, the first few solutions of $J\mathscr{P} = 0$ being

$$\begin{aligned}
&P_{-1n} = Q_{-1n} = 0 \qquad P_{0n} = u_{1n}^{-1} \qquad Q_{0n} = v_{1n}^{-1}\\
&P_{1n} = u_{1n}^{-2}(-u_{0n} + v_{1n}u_{1n+1}^{-1} + v_{1n-1}u_{1n-1}^{-1}) \qquad Q_{1n} = -v_{1n}^{-2}v_{0n} - u_{1n}^{-1}u_{1n+1}^{-1}\\
&P_{2n} = -u_{1n}^{-1}(v_{1n+1}Q_{2n+1} + v_{1n}Q_{2n}) - u_{1n}^{-1}u_{1n}P_{1n} - u_{1n}^{-1}v_{0n}(v_{0n+1}Q_{1n+1} + v_{0n}Q_{1n})\\
&Q_{2n} = v_{0n}^2 v_{1n}^{-3} + (u_{0n}u_{1n+1} + u_{0n+1}u_{1n})u_{1n}^{-2}u_{1n+1}^{-2} - v_{1n}u_{1n}^{-2}u_{1n+1}^{-2}\\
&\qquad\qquad - v_{1n+1}u_{1n+1}^{-2}u_{1n+2}^{-1} - v_{1n-1}u_{1n-1}^{-1}u_{1n}^{-2}u_{1n+1}^{-1}.
\end{aligned} \tag{2.11}$$

From (2.11), we can easily calculate the following equations:

$$\left(\frac{\delta}{\delta u_0}, \frac{\delta}{\delta v_0}, \frac{\delta}{\delta u_1}, \frac{\delta}{\delta v_1}\right) H_{-1} = (P_{-1}, Q_{-1}, P_0, Q_0) \tag{2.12a}$$

$$\left(\frac{\delta}{\delta u_0}, \frac{\delta}{\delta v_0}, \frac{\delta}{\delta u_1}, \frac{\delta}{\delta v_1}\right) H_0 = (P_0, Q_0, P_1, Q_1) \tag{2.12b}$$

where

$$H_{-1}=\ln u_{1n}+\ln v_{1n} \qquad H_0=(v_{0n}v_{1n}^{-1}+u_{0n}u_{1n}^{-1})-v_{1n}u_{1n}u_{1n+1}^{-1}. \quad (2.12c)$$

Thus, by means of Magri's lemma [8, 9], given that $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are compatible Hamiltonian structures (proved in next section), we may deduce from (2.7a) that there exists a sequence of functions H_m, such that $\mathbb{P}^{(m)}=\delta H_m$ for all $m\geqslant -1$. Thus, the flows (2.7a) are tri-Hamiltonian:

$$\boldsymbol{u}_{t_m}=\boldsymbol{B}_0\delta H_{m+2}=\boldsymbol{B}_1\delta H_{m+1}=\boldsymbol{B}_2\delta H_m. \tag{2.13}$$

In the next section, we construct Miura maps and prove the Hamiltonian nature and compatibility of $\boldsymbol{B}_1$.

3. Miura maps and modified systems

Miura maps for the Toda lattice ($u=u_0-\lambda$, $v=v_0$) were given by Kupershmidt [7]. The approach taken by Kupershmidt is a generalization of the factorization method. One of the Miura maps for the Toda lattice reads [7]

$$u_0=\tilde{q}_0+\tilde{q}_1 \qquad v_0=\tilde{q}_1(\boldsymbol{E}\tilde{q}_0). \tag{3.1}$$

Under this map, the Hamiltonian operator of the modified Toda lattice goes to the second Hamiltonian operator of the Toda lattice

$$\tilde{M}'\tilde{J}(\tilde{M}')^{\dagger}=J_0 \tag{3.2}$$

where

$$\tilde{J}=\begin{pmatrix} 0 & \tilde{q}_0(\boldsymbol{I}-\boldsymbol{E}^{-1})\tilde{q}_1 \\ \tilde{q}_1(\boldsymbol{E}-\boldsymbol{I})\tilde{q}_0 & 0 \end{pmatrix}.$$

J_0 is just the operator of (2.4), and $\tilde{M}$ is the Jacobian of the Miura map (3.1).

We now introduce a transformation which takes a constant coefficient operator to $\tilde{J}$, so that the combination of this with Miura map (3.1) yields a new (but equivalent) Miura map, which takes the constant coefficient Hamiltonian operator into J_0.

Under these manipulations, we have

$$u_0=\mathrm{e}^{q_0}+\mathrm{e}^{p_0} \qquad v_0=\mathrm{e}^{q_0}(\boldsymbol{E}\mathrm{e}^{p_0}) \tag{3.3}$$

and it is easy to check

$$\hat{M}'D(\hat{M}')^{\dagger}=J_0 \tag{3.4}$$

where

$$\hat{M}=\hat{M}=\begin{pmatrix} \mathrm{e}^{p_0} & \mathrm{e}^{q_0} \\ \mathrm{e}^{q_0}\boldsymbol{E} & (\boldsymbol{E}\mathrm{e}^{p_0}) \end{pmatrix} \qquad D=\begin{pmatrix} 0 & \boldsymbol{I}-\boldsymbol{E}^{-1} \\ \boldsymbol{E}-\boldsymbol{I} & 0 \end{pmatrix}.$$

In order to extend this Miura Map to our four-component case, we introduce two new variables (q_1, r_1) and linearize the map (3.3):

$$u_1=q_1\,\mathrm{e}^{q_0}+p_1\,\mathrm{e}^{p_0} \qquad v_1=q_1\,\mathrm{e}^{q_0}(\boldsymbol{E}\ \mathrm{e}^{p_0})+\mathrm{e}^{q_0}(\boldsymbol{E}\ \mathrm{e}^{p_0})(\boldsymbol{E}p_1). \tag{3.5}$$

It is easy to check that under the map M: $(q_0, p_0, q_1, p_1)\to(u_0, v_0, u_1, v_1)$ given by (3.3) and (3.5) the Hamiltonian structure $\boldsymbol{B}_2$ of (2.7b) is the image $\boldsymbol{B}_2^2$, given by:

$$M'\boldsymbol{B}_2^2(M')^{\dagger}=\boldsymbol{B}_2 \tag{3.6}$$

where

$$\boldsymbol{B}_2=\begin{pmatrix}0 & J_0\\ J_0 & J_1\end{pmatrix}\qquad \boldsymbol{B}_2^2=\begin{pmatrix}0 & D\\ D & 0\end{pmatrix}$$

and M' is the Jacobian of (3.3) and (3.5) with:

$$M'=\begin{pmatrix}m_0 & 0\\ m_1 & m_0\end{pmatrix}\qquad m_0=\begin{pmatrix}\mathrm{e}^{p_0} & \mathrm{e}^{q_0}\\ \mathrm{e}^{q_0}\boldsymbol{E} & (\boldsymbol{E}\,\mathrm{e}^{p_0})\end{pmatrix}$$

$$m_1=\begin{pmatrix}\mathrm{e}^{p_0} & \mathrm{e}^{q_0}q_1\\ q_1\,\mathrm{e}^{q_0}(\boldsymbol{E}\,\mathrm{e}^{p_0})\boldsymbol{E}+\mathrm{e}^{q_0}(\boldsymbol{E}p_1)(\boldsymbol{E}\,\mathrm{e}^{p_0})\boldsymbol{E} & q_1\,\mathrm{e}^{q_0}(\boldsymbol{E}\,\mathrm{e}^{p_0})+\mathrm{e}^{q_0}(\boldsymbol{E}\,\mathrm{e}^{p_0})(\boldsymbol{E}p_1)\end{pmatrix}.$$

As was observed for the third-order operator in [3], the map M can be decomposed as $M=M_1\circ M_2$ with

$$M_1'=\begin{pmatrix}m_0 & 0\\ 0 & I\end{pmatrix}\qquad M_2'=\begin{pmatrix}I & 0\\ m_1 & m_0\end{pmatrix}\tag{3.7}$$

corresponding to first performing (3.5), keeping (p_0, q_0) fixed, and then keeping (p_1, q_1) fixed while changing (p_0, q_0) by (3.3). This means that we have an intermediate integrable system. Indeed, the remarkable picture appears again (see [1] and [3]):

$$\begin{array}{lll}
(\boldsymbol{u}_0,\boldsymbol{u}_1) \longleftarrow & (\boldsymbol{p}_0,\boldsymbol{u}_1) \longleftarrow & (\boldsymbol{p}_0,\boldsymbol{p}_1)\\
\boldsymbol{B}_2=\boldsymbol{B}_2^0 \longleftarrow & \boldsymbol{B}_2^1 \longleftarrow & \boldsymbol{B}_2^2=\boldsymbol{B}\\
\boldsymbol{B}_1=\boldsymbol{B}_1^0 \longleftarrow & \boldsymbol{B}_1^1 & \\
\boldsymbol{B}_0=\boldsymbol{B}_0^0 & &
\end{array}$$

Figure 1

where $\boldsymbol{u}_k=(u_k, v_k)$, $\boldsymbol{p}_k=(q_k, p_k)$, and

$$\boldsymbol{B}_2^1=\begin{pmatrix}0 & Dm_0^\dagger\\ m_0D & \hat{\mathscr{J}}_1\end{pmatrix}\qquad \boldsymbol{B}_1^1=\begin{pmatrix}D & 0\\ 0 & -\bar{J}_2\end{pmatrix}\qquad \hat{\mathscr{J}}_1=m_0Dm_1^\dagger+m_1Dm_0^\dagger.\tag{3.8}$$

In (3.8), $\bar{J}_2$ differs from J_2 of (2.4) by the substitution of (3.3) for $\boldsymbol{u}_0$. It is easy to check that $\boldsymbol{B}_2^1$ and $\boldsymbol{B}_1^1$ depend only on $(\boldsymbol{p}_0, \boldsymbol{u}_0)$. This is the result of the mysterious Miura maps. In [11], Wilson presents a detailed analysis of this point in the context of the KdV from an algebraic point of view.

Remarks. (1) We used the same approach here as in [3]. However, in the energy-dependent Schrödinger and the third-order cases, one is able to derive the related Miura maps by means of gauge transformations [12], whereas, it is not clear that one is able to do this here.

(2) The Toda lattice is known to be a tri-Hamiltonian system, i.e. there exist three local Hamiltonian operators for it. Based on the results for KdV [1] and the Boussinesq equation [3], one may expect that the four-component system possesses four local Hamiltonian operators. But this turns out not to be the case.

We can prove the compatibility of $\boldsymbol{B}_0, \boldsymbol{B}_1, \boldsymbol{B}_2$ by the spectral parameter shift technique as in [3]. This also shows that the Hamiltonian nature of $\boldsymbol{B}_0, \boldsymbol{B}_1$ follows from that of $\boldsymbol{B}_2$. The argument is identical to that in [3].

Now we need to prove the Hamiltonian nature of $\boldsymbol{B}_1^1, \boldsymbol{B}_2^1$. The relation between $\boldsymbol{B}_1^1$ and $\boldsymbol{B}_2^2$ $(=D)$ immediately verifies that $\boldsymbol{B}_1^1$ is a Hamiltonian operator. As far as $\boldsymbol{B}_2^1$ is concerned, we are able to reduce it to a constant coefficient operator by a simple coordinate transformation because of the special form of $\bar{J}_2$.

Thus, we have proven the assertions made in the above section. The strategy we employed is the following: first we presented the Miura maps so that we have three hierarchies of completely integrable Hamiltonian systems. Then, by means of the Miura maps we proved the Hamiltonian nature of the operators involved in figure 1 and the relevant compatibility.

Now it is appropriate to give some explicit examples.

Example. The first non-trivial flow is

$$\boldsymbol{u}_t = \boldsymbol{B}_1 \delta H_0 = \boldsymbol{B}_2 \delta H_{-1} \tag{3.9}$$

given explicitly as:

$$\begin{aligned}
u_{0t} &= v_0 (u_1^{(1)})^{-1} = v_0^{(-1)} (u_1^{(-1)})^{-1} + u_0 v_0 v_1^{-1} = u_0 v_0^{(-1)} (v_1^{(-1)})^{-1} \\
v_{0t} &= v_0 (u_1^{(1)})^{-1} u_0^{(1)} - v_0 u_0 u_1^{-1} + v_0 v_0^{(1)} (v_1^{(1)})^{-1} - v_0 v_0^{(-1)} (v_1^{(-1)})^{-1} \\
u_{1t} &= u_1 \{ v_1^{-1} v_0 - v_1 u_1^{-1} (u_1^{(1)})^{-1} - v_0^{(-1)} (v_1^{(-1)})^{-1} - v_1^{(-1)} (u_1^{(-1)})^{-1} u_1^{-1} \} \\
v_{1t} &= v_1 \{ -(u_1^{(1)})^{-1} u_0^{(1)} + u_0 u_1^{-1} - v_0^{(1)} (v_1^{(1)})^{-1} + v_0^{(-1)} (v_1^{(-1)})^{-1} \}.
\end{aligned}$$

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References

[1] Antonowicz M and Fordy A P 1989 *Commun. Math. Phys.* **124** 465–86
[2] Antonowicz M and Fordy A P 1987 *Physica* **28D** 345–58
[3] Antonowicz M, Fordy A P and Liu Q P 1991 *Nonlinearity* 4 669–84
[4] Tu G Z 1990 *J. Phys. A: Math. Gen.* **23** 3902–22
[5] Ragnisco O 1988 private communication
[6] Ragnisco O and Santini P M 1988 *J. Math. Phys.* **29** 1593–603
[7] Kuperschmidt B A 1985 *Astérisque* **123**
[8] Magri F 1978 *J. Math. Phys.* **19** 1156–62
[9] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[10] Schilling R J 1989 *J. Math. Phys.* **30** 1487–501
[11] Wilson G 1988 *Phys. Lett.* **132A** 445–50
[12] Liu Q P 1992 PhD Thesis Leeds University